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SOUND PROPAGATION FORMULATION FOR 3-D RAY-TRACE COMPUTATIONS SP--ETC(U)
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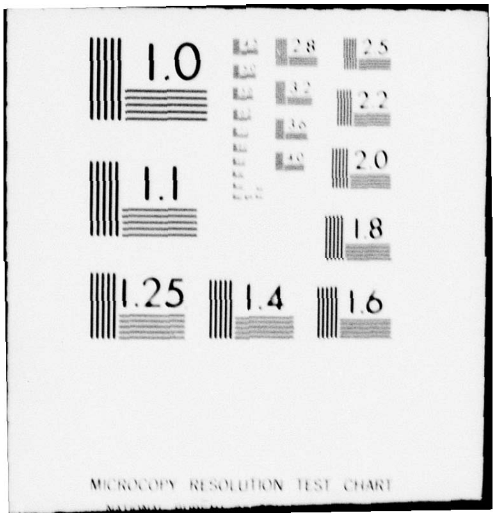
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METEOROLOGY INTERNATIONAL INCORPORATED

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LEVEL III

6

SOUND PROPAGATION
FORMULATION FOR 3-D RAY-TRACE COMPUTATIONS
SPHERICAL EARTH.

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Problem:

Given the speed-of-sound distribution in the sea the problem is to calculate the trace of a sound ray which leaves[1] a prescribed location in a prescribed direction. The formulation includes the curvature-of-the-earth factor and other curvature terms arising in the selection of horizontal coordinates.

Part A: Formulation of the Equations

1. The Governing Vector Equation

The ray is directed normal to, and in the direction of advancement of, the wave front. We assign the unit vector $\underset{\sim}{t}$[2] to this direction. At a point in the progression of the sound ray the unit vector $\underset{\sim}{t}$ and the vector ∇c, being the local ascendent of the sound speed, together generally define a plane. This plane is the plane of Fig. 1.

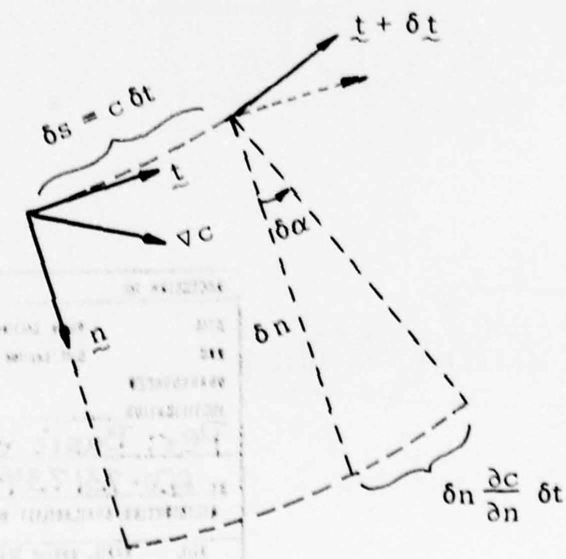

Fig. 1 Refraction of the Ray

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[1] or arrives at (The integration may also be performed backwards in time.)
[2] Geometric vectors are denoted by curl underscore.

The component of ∇c which is normal to $\underset{\sim}{t}$, and hence lies in the wave front, may be expressed by

$$- \underset{\sim}{t} \times (\underset{\sim}{t} \times \nabla c) \equiv \nabla c - \underset{\sim}{t}\, \underset{\sim}{t} \cdot \nabla c \equiv \underset{\sim}{n}\, \underset{\sim}{n} \cdot \nabla c \, . \tag{1}$$

It is this component of ∇c which causes the differential progression of the wave front and the associated bending of the ray.

Let the parameter s be a linear measure in the progression of the ray. In time increment δt the length increment traversed is

$$\delta s = c\, \delta t \, . \tag{2}$$

In terms of the geometry of Fig. 1 the bending of the ray may be expressed by

$$\frac{\delta \alpha}{\delta t} = \frac{\partial c}{\partial n} \, . \tag{3}$$

The corresponding governing equation, expressed in vector form, is

$$\frac{D\underset{\sim}{t}}{Dt} = \underset{\sim}{t} \times (\underset{\sim}{t} \times \nabla c) \, . \tag{4}$$

The differential operator, D/Dt, implies differentiation in time moving with the ray. Equation (4) readily follows from Eq. (3) in that $\delta \underset{\sim}{t} = - \underset{\sim}{n}\, \delta \alpha$. The change in a <u>unit</u> vector is always directed normal to the vector.

If the speed of sound is relatively constant in time over the period of propagation of the ray we may choose to integrate along path length rather than in terms of time. In this case Eq. (4) may be expressed by

$$\frac{D\underset{\sim}{t}}{Ds} = \underset{\sim}{t} \times (\underset{\sim}{t} \times \nabla K) , \tag{5}$$

where

$$K \equiv \ln c . \tag{6}$$

2. Vertical and Horizontal Components

We proceed by relating the vector equation, Eq. (5), to the natural reference orientations of horizontal and vertical. The unit vector $\underset{\sim}{t}$, defined as before as the local direction of the ray, and the unit vector $\underset{\sim}{k}$, defined as the direction of local gravity, together generally define a vertical plane. This plane is the plane of Fig. 2.

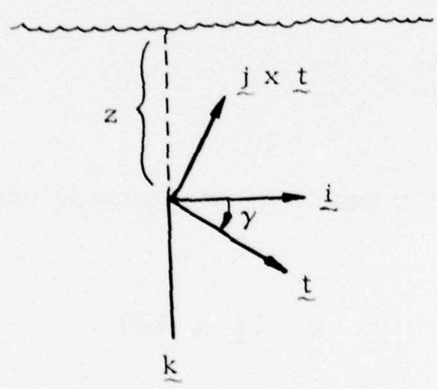

Fig. 2 Vertical and Horizontal References

The horizontal component of $\underset{\sim}{t}$ defines the orientation of the unit vector $\underset{\sim}{i}$. The unit vector $\underset{\sim}{j}$ completes a right-handed triple:

$$\underset{\sim}{k} \times \underset{\sim}{i} = \underset{\sim}{j} . \tag{7}$$

And $\mathbf{j}$ points out of the plane of Fig. 2, toward the reader. Whereas the vector $\mathbf{k}$ is locally defined (i.e. a function of position only) the vectors $\mathbf{j}$ and $\mathbf{i}$ depend on the particular ray in that $\mathbf{i}$ is parallel to the horizontal component of $\mathbf{t}$. The angle γ is defined as shown in Fig. 2. The parameter Z denotes the depth below sea level of the location.

Directional change in the <u>unit</u> vector $\mathbf{t}$ along the ray path, s, involves only two components, and these lie in the plane of the wave front. The plane of the wave front may be defined by the vector pair $\mathbf{j}$ and $\mathbf{j} \times \mathbf{t}$.

The vectors $\mathbf{t}$ and $\mathbf{j} \times \mathbf{t}$ may be expressed by

$$\mathbf{t} = \mathbf{i} \cos\gamma + \mathbf{k} \sin\gamma,$$

$$\mathbf{j} \times \mathbf{t} = -\mathbf{k} \cos\gamma + \mathbf{i} \sin\gamma. \tag{9}$$

We proceed by developing the left-hand side of Eq. (5):

$$\begin{aligned}
\frac{D\mathbf{t}}{Ds} &= \frac{D}{Ds}\left(\mathbf{i}\cos\gamma + \mathbf{k}\sin\gamma\right) \\
&= \left(-\mathbf{i}\sin\gamma + \mathbf{k}\cos\gamma\right)\frac{D\gamma}{Ds} + \cos\gamma\,\frac{D\mathbf{i}}{Ds} + \sin\gamma\,\frac{D\mathbf{k}}{Ds} \\
&= -(\mathbf{j}\times\mathbf{t})\frac{D\gamma}{Ds} + \cos\gamma\left(\mathbf{j}\,\mathbf{j}\cdot\frac{D\mathbf{i}}{Ds} + \mathbf{k}\,\mathbf{k}\cdot\frac{D\mathbf{i}}{Ds}\right) + \sin\gamma\,\frac{D\mathbf{k}}{Ds} \\
&= -(\mathbf{j}\times\mathbf{t})\frac{D\gamma}{Ds} + \cos\gamma\,\mathbf{j}\,\mathbf{j}\cdot\frac{D\mathbf{i}}{Ds} + \left(-\cos\gamma\,\mathbf{k}\,\mathbf{i}\cdot + \sin\gamma\right)\frac{D\mathbf{k}}{Ds}.
\end{aligned} \tag{10}$$

We approximate the earth-curvature factor by

$$\frac{D\mathbf{k}}{Ds} = -\mathbf{i}\,\frac{\cos\gamma}{R - z} \tag{11}$$

where R is the local radius of curvature of the earth. Substitution of Eq. (11) in Eq. (10) yields:

$$\frac{D\underset{\sim}{t}}{Ds} = -(\underset{\sim}{j} \times \underset{\sim}{t})\frac{D\gamma}{Ds} + \cos\gamma\, \underset{\sim}{j}\, \underset{\sim}{j} \cdot \frac{D\underset{\sim}{i}}{Ds} - (\underset{\sim}{j} \times \underset{\sim}{t})\frac{\cos\gamma}{R-z} \,. \qquad (12)$$

Making use of Eq. (12) we now take the $(\underset{\sim}{j} \times \underset{\sim}{t})$ component of Eq. (5):

$$\frac{D\gamma}{Ds} = -\frac{\cos\gamma}{R-z} - (\underset{\sim}{j} \times \underset{\sim}{t}) \cdot \underset{\sim}{t} \times (\underset{\sim}{t} \times \nabla K)$$

$$= -\frac{\cos\gamma}{R-z} + (\underset{\sim}{j} \times \underset{\sim}{t}) \cdot \nabla K$$

$$= -\frac{\cos\gamma}{R-z} - \cos\gamma \frac{\partial K}{\partial z} + \sin\gamma\, \underset{\sim}{i} \cdot \nabla K \,. \qquad (13)$$

And the $\underset{\sim}{j}$ component:

$$\cos\gamma\, \underset{\sim}{j} \cdot \frac{D\underset{\sim}{i}}{Ds} = \underset{\sim}{j} \cdot \underset{\sim}{t} \times (\underset{\sim}{t} \times \nabla K) = -\underset{\sim}{j} \cdot \nabla K \,. \qquad (14)$$

Equations (13) and (14) are the desired pair of components of the governing equation, Eq. (5).

Provided the ray does not become vertical, we may also use a linear measure, x, along $\underset{\sim}{i}$, as the independent argument for the integration:

$$Dx = Ds \cos\gamma \,. \qquad (15)$$

We may also introduce the linear measure y along $\underset{\sim}{j}$. With these definitions, Eqs. (13) and (14) may be expressed by

$$\frac{D\gamma}{Dx} = - \frac{1}{R-z} - \frac{\partial K}{\partial z} + \tan\gamma \frac{\partial K}{\partial x} , \tag{16}$$

$$\underset{\sim}{j} \cdot \frac{D\underset{\sim}{i}}{D x} = - \sec^2 \gamma \frac{\partial K}{\partial y} . \tag{17}$$

3. Horizontal Coordinates

Let ϵ and ω be a suitable pair of orthogonal coordinates for the horizontal expanse. The ascendents define the corresponding unit vectors $\underset{\sim}{I}$ and $\underset{\sim}{J}$:

$$\nabla\epsilon \equiv E\underset{\sim}{I} ,$$

$$\nabla\omega \equiv W\underset{\sim}{J} . \tag{19}$$

We choose that $\underset{\sim}{I}$ and $\underset{\sim}{J}$ form a left-handed system with $\underset{\sim}{k}$:

$$\underset{\sim}{I} \times \underset{\sim}{J} = -\underset{\sim}{k} . \tag{20}$$

At an arbitrary location the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$, defined earlier, form angle β with $\underset{\sim}{I}$ as shown in Fig. 3.

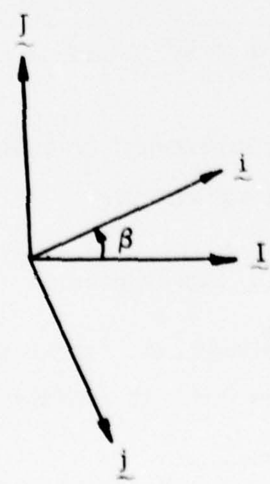

Fig. 3 Definition of the Angle β

We proceed as follows:

$$\underset{\sim}{j} \cdot \frac{D\underset{\sim}{i}}{Dx} = \underset{\sim}{j} \cdot \frac{D}{Dx} (\cos\beta \underset{\sim}{I} + \sin\beta \underset{\sim}{J})$$

$$= \underset{\sim}{j} \cdot (-\sin\beta \underset{\sim}{I} + \cos\beta \underset{\sim}{J}) \frac{D\beta}{Dx} + \cos\beta \frac{D\underset{\sim}{i}}{Dx} + \sin\beta \frac{D\underset{\sim}{J}}{Dx}$$

$$= -\frac{D\beta}{Dx} + (\cos\beta \underset{\sim}{j} - \sin\beta \underset{\sim}{j} \times \underset{\sim}{k}) \cdot \frac{D\underset{\sim}{I}}{Dx}$$

$$= -\frac{D\beta}{Dx} - \underset{\sim}{J} \cdot \frac{D\underset{\sim}{I}}{Dx}$$

$$= -\frac{D\beta}{Dx} - \underset{\sim}{J} \cdot (\underset{\sim}{i} \cdot \nabla \underset{\sim}{I}) . \tag{21}$$

Substitution in Eq. (17) yields

$$\frac{D\beta}{Dx} = Q + \sec^2\gamma \quad \partial K/\partial y , \tag{22}$$

where

$$Q \equiv -\underset{\sim}{J} \cdot (\underset{\sim}{i} \cdot \nabla \underset{\sim}{I}) \tag{23}$$

is due to the curvature of the horizontal coordinates. Equation (16) is the other of our pair of governing equations.

4. Longitude and Latitude Coordinates

Longitude, θ, and latitude, φ, form a pair of orthogonal coordinates[3]. The ascendents define the unit vectors:

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[3] Longitude is measured <u>east</u> of Greenwich only.

$$\nabla\theta = \underset{\sim}{I} / (R-z) \cos\varphi \tag{24}$$

$$\nabla\varphi = \underset{\sim}{J} / (R-z) \, . \tag{25}$$

These expressions are singular at the poles. The curvature term, Eq. (23), develops as follows:

$$Q = - \underset{\sim}{J} \cdot \left\{ \cos\beta \, \underset{\sim}{I} + \sin\beta \, \underset{\sim}{J} \right\} \cdot \nabla \underset{\sim}{I}$$

$$= - \underset{\sim}{J} \cdot \left\{ \cos\beta \, \underset{\sim}{I} \cdot \nabla \underset{\sim}{I} + \sin\beta \, \underset{\sim}{J} \cdot \nabla \underset{\sim}{I} \right\}$$

$$= - \underset{\sim}{J} \cdot \left\{ \cos\beta \, \underset{\sim}{J} \, \frac{\tan\varphi}{R-z} \right\} = - \cos\beta \, \tan\varphi / R-z \, . \tag{26}$$

The governing equations for the ray trace may be written:

$$\frac{D\gamma}{Dx} = - \frac{1}{R-z} - \frac{\partial K}{\partial z} + \tan\gamma \, (\cos\beta \, \underset{\sim}{I} + \sin\beta \, \underset{\sim}{J}) \cdot \nabla K \tag{27}$$

$$\frac{D\beta}{Dx} = - \frac{\cos\beta \tan\varphi}{R-z} + \sec^2\gamma \, (\sin\beta \, \underset{\sim}{I} - \cos\beta \, \underset{\sim}{J}) \cdot \nabla K \, . \tag{28}$$

The directional derivatives may be expressed by

$$\underset{\sim}{I} \cdot \nabla K = \frac{1}{(R-z)\cos\varphi} \frac{\partial K}{\partial\theta} \tag{29}$$

$$\underset{\sim}{J} \cdot \nabla K = \frac{1}{R-z} \frac{\partial K}{\partial\varphi} \, . \tag{30}$$

Also required, as increments of integration, are

$$\delta\theta = \frac{\cos\beta}{(R-z)\cos\varphi} \, \delta x \, , \tag{31}$$

$$\delta\varphi = \frac{\sin\beta}{R-z}\,\delta x\,, \tag{32}$$

where δx is true horizontal trace-length increment.

5. Cartesian Coordinates in a Polar Stereographic Projection

The coordinates

$$X \equiv L\cos\theta \tag{33}$$

$$Y \equiv L\sin\theta\,, \tag{34}$$

where

$$L \equiv (R-z)\,M\cos\varphi \tag{35}$$

$$M \equiv \frac{1+\sin\varphi_o}{1+\sin\varphi} \tag{36}$$

map into Cartesian coordinates in a polar stereographic projection, true at latitude φ_o. The coordinate origin is at the pole.

The ascendents define the unit vectors:

$$\nabla X \equiv M\,\underset{\sim}{I}\,, \tag{37}$$

$$\nabla Y \equiv M\,\underset{\sim}{J}\,. \tag{38}$$

The curvature term, Eq. (23), transforms as follows:

$$Q = -\underset{\sim}{J} \cdot \left\{ \cos\beta\, \underset{\sim}{I} \cdot \nabla \underset{\sim}{I} \;+\; \sin\beta\, \underset{\sim}{J} \cdot \nabla \underset{\sim}{I} \right\}$$

$$= \left\{ -\cos\beta \sin\theta + \sin\beta \cos\theta \right\} \frac{1 - \sin\varphi}{(R-z)\cos\varphi} \;. \qquad (39)$$

This can be shown by expressing the unit vectors in terms of the polar vectors, Eqs. (24) and (25). Elimination of θ and φ from Eq. (39), using Eqs. (33) through (36), leads to

$$Q = \frac{\sin\beta X - \cos\beta Y}{(R-z)^2 \;(1 + \sin\varphi_o)} \;. \qquad (40)$$

The governing equations for the ray trace may be written:

$$\frac{D\gamma}{Dx} = -\frac{1}{R-z} - \frac{\partial K}{\partial z} + \tan\gamma \left\{ \cos\beta \; M \frac{\partial K}{\partial X} + \sin\beta \; M \frac{\partial K}{\partial Y} \right\} \qquad (41)$$

$$\frac{D\beta}{Dx} = \frac{\sin\beta\, X - \cos\beta Y}{(R-z)^2 \;(1 + \sin\varphi_o)} + \sec^2\gamma \left\{ \sin\beta \; M \frac{\partial K}{\partial X} - \cos\beta M \frac{\partial K}{\partial Y} \right\} . \qquad (42)$$

6. Integration Formulae

We choose to carry on with the coordinates introduced in Section 5: Cartesian coordinates in a polar stereographic projection.

The governing equations, Eqs. (41) and (42), may be developed for numerical integration by introducing finite increments:

$$\delta\gamma = -\left\{ \frac{1}{R-z} + \frac{\partial K}{\partial z} \right\} \delta x + \tan\gamma \left\{ \frac{\partial K}{\partial X} \;\delta X + \frac{\partial K}{\partial Y} \;\delta Y \right\} \qquad (43)$$

$$= M \frac{X\,\delta Y - Y\,\delta X}{(R-z)^2 \;(1 + \sin\varphi_o)} + \sec^2\gamma \left\{ \frac{\partial K}{\partial X} \,\delta Y - \frac{\partial K}{\partial Y} \;\delta X \right\} . \qquad (44)$$

To these we may add:

$$\delta X = M \cos \beta \, \delta x \tag{45}$$

$$\delta Y = M \sin \beta \, \delta x \tag{46}$$

$$\delta Z = \tan \gamma \, \delta x \tag{47}$$

For integrations in which the trace is permitted to become vertical we must reintroduce

$$\delta x = \cos \gamma \ \delta s \ , \tag{48}$$

and use δs, the linear progression of the ray, as the argument of integration.

For integration with time as the argument we reintroduce

$$\delta s = c \, \delta t \ . \tag{49}$$

In order to succintly handle surface and bottom reflections the argument may be geared to depth increment δz, unless an upper bound on δx is exceeded in which case δz must be reduced.

At the surface, and at the bottom, reflections are performed by reversing the sign of γ. At the bottom this involves the approximation that the bottom is flat.

Part B: *A Scheme for the Numerical Integration*

We herewith develop a proposed integration scheme based on the geometric visualization of refraction and the adjustments for earth and coordinate curvatures. Equations (41) and (42) express the ray propagation as differential equations in terms of Cartesian coordinates in a polar stereographic projection. In order to visualize the geometry we rewrite these equations as follows:

<u>curvature term</u> <u>refraction term</u>

$$\delta\gamma = \frac{-\delta x}{R-z} + \underset{\sim}{j} \times \underset{\sim}{t} \cdot \nabla K \quad \delta s \tag{50}$$

$$\delta\beta = \frac{\sin\beta X - \cos\beta Y}{(R-z)^2\ (1+\sin\varphi_o)}\ \delta x + \sec\gamma\ \underset{\sim}{j} \cdot \nabla K \quad \delta s\ . \tag{51}$$

The vectors $\underset{\sim}{j} \times \underset{\sim}{t}$ and $\underset{\sim}{j}$ may be recollected from Fig. 2: The vector $\underset{\sim}{j} \times \underset{\sim}{t}$ lies in the vertical plane and is normal to the ray. The vector $\underset{\sim}{j}$ lies in the horizontal plane and is normal to the ray.

A segment of the ray, in which segment the refraction terms are taken as constant, describes a circular arc in space. We propose to integrate the ray along <u>chords</u> for such arc segments. <u>Representative</u> values for the refraction terms are each numerically evaluated for the segment.

Along a chord the curvature terms may also be considered steady. This is apparent for the term $-\delta x/R-z$ if we neglect the minor local variation of R-z. For the other curvature term we introduce the argument along the chord increment δs. That is,

$$X = X_o + \cos\beta\ \gamma \tag{52}$$

$$Y = Y_o + \sin\beta\ \gamma\ . \tag{53}$$

Substitution in the coordinate curvature term yield

$$\frac{\sin\beta X - \cos\beta Y}{(R-z)^2 \quad (1 + \sin\varphi_o)} = \frac{\sin\beta X_o - \cos\beta Y_o}{(R-z)^2 \quad (1 + \sin\varphi_o)} \tag{54}$$

and shows that this term may also be considered steady along a chord.

We assume that K, the natural logarithm of the speed of sound, is given at an array of grid points in the polar stereographic projection, for each of a number of depth levels.

We shall describe the integration in terms of the incremental step from position X_τ, Y_τ, z_τ, at which point the ray is directed at angles γ_τ, β_τ, to the next position where these values are indicated by subscripts $\tau + 1$.

Let $\gamma_{\tau+1}^{(R)}$, $\beta_{\tau+1}^{(R)}$ be the R-th estimate of the angles with which the ray is directed at position $\tau + 1$. The first guess of these angles may be approximated by

$$\gamma_{\tau+1}^{(o)} = \gamma_\tau , \tag{55}$$

$$\beta_{\tau+1}^{(o)} = \beta_\tau . \tag{56}$$

Linear extrapolation from $\tau - 1$ and τ values may yield a better first guess. The chord is directed from position τ with the angles

$$\gamma^{(R)} = \frac{1}{2} (\gamma_\tau + \gamma_{\tau+1}^{(R)}) , \tag{57}$$

$$\beta^{(R)} = \frac{1}{2} (\beta_\tau + \beta_{\tau+1}^{(R)}) . \tag{58}$$

Choosing a chord length, δs , the R-th estimate of the ray's advance is given by

$$\delta X^{(R)} = M \cos \beta^{(R)} \cos \gamma^{(R)} \delta s , \tag{59}$$

$$\delta Y^{(R)} = M \sin \beta^{(R)} \cos \gamma^{(R)} \delta s , \tag{60}$$

$$\delta z^{(R)} = \sin \gamma^{(R)} \delta s . \tag{61}$$

where M is an appropriately centered value of the mapping factor, Eq. (36). We also require

$$\delta x^{(R)} = \cos \gamma^{(R)} \delta s . \tag{62}$$

We must now calculate the bending experienced by the ray, over this increment, as expressed by Eqs. (50) and (51).

The curvature terms are evaluated by

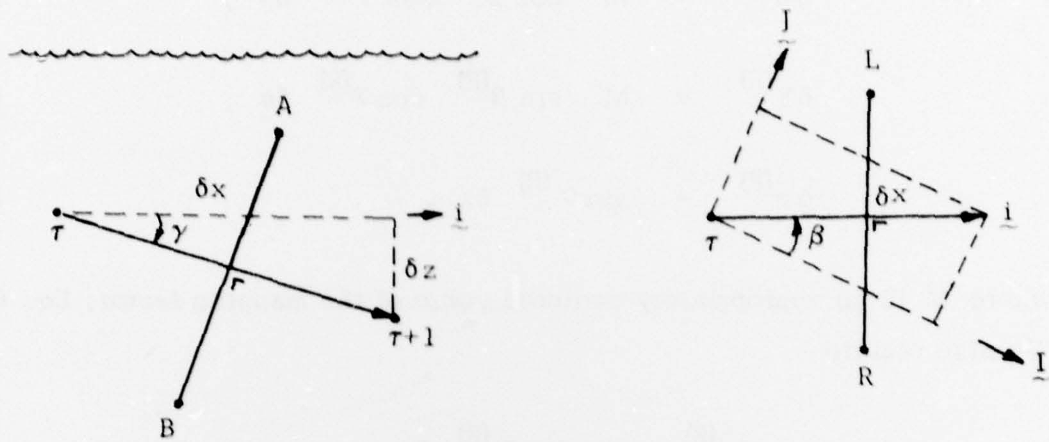
$$\frac{\delta x^{(R)}}{R-z} \tag{63}$$

$$\frac{\sin \beta^{(R)} X_T - \cos \beta^{(R)} Y_T}{(R-z)^2} \quad \frac{\delta x^{(R)}}{(1 + \sin \varphi_o)} \tag{64}$$

where an appropriately centered value of R-z is used.

For evaluating the amount of refraction we require <u>representative</u> values of the refraction terms. We propose that this be done by finite-difference evaluation of the required directional derivatives of K. The space increment for evaluating these should be commensurate with the

length of the chord; in fact we propose that it be made equal to δs to give the desired representativeness.



vertical plane horizontal plane

Fig. 4 Positions A, B, R and L for Evaluating the Directional Derivatives of K

The locations at which K should be interpolated are indicated in Fig. 4. In the vertical plane the positions are A, above, and B, below; their separation is δs, forming a cross with the chord. In the horizontal plane the positions are R, right, and L, left; their separation is δs, forming a cross with the chord. The R-th estimates of these positions are:

$$X_A^{(R)} = X_o + \frac{1}{2}\left(\cos\gamma^{(R)} + \sin\gamma^{(R)}\right)\cos\beta^{(R)}\, M\,\delta s$$

$$Y_A^{(R)} = Y_o + \frac{1}{2}\left(\cos\gamma^{(R)} + \sin\gamma^{(R)}\right)\sin\beta^{(R)}\, M\,\delta s$$

$$z_A^{(R)} = z_o + \frac{1}{2}\left(\sin\gamma^{(R)} - \cos\gamma^{(R)}\right)\delta s$$

$$X_B^{(R)} = X_o + \frac{1}{2} (\cos \gamma^{(R)} - \sin \gamma^{(R)}) \cos \beta^{(R)} \; M \, \delta s$$

$$Y_B^{(R)} = Y_o + \frac{1}{2} (\cos \gamma^{(R)} - \sin \gamma^{(R)}) \sin \beta^{(R)} \; M \, \delta s$$

$$z_B^{(R)} = z_o + \frac{1}{2} (\sin \gamma^{(R)} + \cos \gamma^{(R)}) \; \delta s$$

$$X_R^{(R)} = X_o + \frac{1}{2} (\cos \gamma^{(R)} \cos \beta^{(R)} + \sin \beta^{(R)}) \; M \, \delta s$$

$$Y_R^{(R)} = Y_o + \frac{1}{2} (\cos \gamma^{(R)} \sin \beta^{(R)} - \cos \beta^{(R)}) \; M \, \delta s$$

$$z_R^{(R)} = z_o + \frac{1}{2} \sin \gamma^{(R)} \; \delta s$$

$$X_L^{(R)} = X_o + \frac{1}{2} (\cos \gamma^{(R)} \cos \beta^{(R)} - \sin \beta^{(R)}) \; M \, \delta s$$

$$Y_L^{(R)} = Y_o + \frac{1}{2} (\cos \gamma^{(R)} \sin \beta^{(R)} + \cos \beta^{(R)}) \; M \, \delta s$$

$$z_L^{(R)} = z_o + \frac{1}{2} \sin \gamma^{(R)} \; \delta s \; . \tag{65}$$

Let the values of K interpolated at these locations be denoted by

$$K_A^{(R)} \; , \quad K_B^{(R)} \; , \quad K_R^{(R)} \quad \text{and} \quad K_L^{(R)} \; . \tag{66}$$

The refraction terms are accordingly given by

$$(\underset{\sim}{j} \times \underset{\sim}{t} \cdot \nabla K \; \delta s)^{(R)} = K_A^{(R)} - K_B^{(R)} \tag{67}$$

$$\sec \gamma^{(R)} \; (\underset{\sim}{j} \cdot \nabla K \; \delta s)^{(R)} = \sec \gamma^{(R)} \; (K_R^{(R)} - K_L^{(R)}) \tag{68}$$

The R-th estimate of the bending of the ray in the coordinate system is given by

$$\delta \gamma^{(R)} = \frac{\delta x^{(R)}}{R-z} + K_A^{(R)} - K_B^{(R)} , \tag{69}$$

$$\delta\beta^{(R)} = \frac{\sin \beta^{(R)} X_o - \cos \beta^{(R)} Y_o}{(R-z)^2 \ (1 + \sin \varphi_o)} \delta x^{(R)} + \sec \gamma^{(R)} (K_R^{(R)} - K_L^{(R)}) . \tag{70}$$

These give the R+1 estimate of the angles with which the ray is directed at position $\tau+1$:

$$\gamma_{\tau+1}^{(R+1)} = \gamma_\tau + \delta \gamma^{(R)} , \tag{71}$$

$$\beta_{\tau+1}^{(R+1)} = \beta_\tau + \delta\beta^{(R)} . \tag{72}$$

With these new estimates the R-1 cycle is performed beginning again with Eq. (57). The cycle is repeated until convergence is attained (i.e. within some finite tolerance). Upon such convergence the desired chord has been found, the incremental step is completed, and the ray moves to position $\tau+1$.

It should be noted that in evaluating the coordinate curvature term, Eq. (64), the origin for X and Y is at the pole. Also, the angle β is the angle made by the ray with the X axis; that is,

$$\cos \beta = \underset{\sim}{i} \cdot \underset{\sim}{I} . \tag{73}$$

In the suggested scheme, the chord length, δs, may be chosen for each step. We feel that the optimum choice for δs would be such that convergence is attained at R=3. However, it may be worth experimenting with δs to see how the convergence is affected.

Reflection of the ray, at the sea surface and at the bottom also influences the choice of δs. Let the distance to the boundary being approached by the ray be

$$\delta z^* = \left| z_* - z_T \right| \tag{74}$$

where z_* is zero, or the depth, h. If

$$\left| \delta z^{(R)} \right| > \delta z^*$$

then

$$\delta s^{(R+1)} = \frac{\delta z^*}{\sin \gamma^{(R)}} \tag{75}$$

should be used as chord length in the next cycle, so that the step takes the ray exactly to the boundary. At this juncture, γ is replaced by $-\gamma$, assuming the bottom is horizontal.

Another contingency must be provided for. In approaching the upper or lower boundary, one or the other of the A, above, or B, below, locations may fall beyond the boundary. Provision should be made so that the interpolation of K will yield a suitable extrapolated value in such cases.

Flow Diagram in Terms of Operations

Position: $X_\tau \; Y_\tau \; Z_\tau$

Angles of Ray: $\gamma_\tau \; \beta_\tau$

Set estimates: Eqs. (55) and (56)

Compute chord angles: Eqs. (57) and (58)

Choose δs

Is δz too large? No; proceed with δs.

Yes; replace δs according to Eq. (75).

Compute curvature terms: Eqs. (63) and (64)

Compute locations A, B, R, L: Eq. (65)

Interpolate K at A, B, R and L

Compute refraction: Eqs. (67) and (68)

Compute total bending: Eqs. (69) and (70)

Compute $\gamma_{\tau+1}^{(R+1)}$ and $\beta_{\tau+1}^{(R+1)}$: Eqs. (71) and (72)

Test angles for convergence:

No; recycle

Yes; advance step (including any reflection) to $\tau+1$ position and angles

The program requires an interpolation subroutine for interpolating to specific locations in the three-dimensional distribution of $K \equiv \ln c$, where c is the speed of sound. Computationally, the program requires zero-order continuity of the interpolation; hence even linear interpolation between grid-point values could be used. However in the interests of better accuracy higher-order interpolation is preferable.

The three-dimensional distribution of K may be expressed by the two-dimensional distributions for a number of discrete depth levels. These depth levels should be concentrated where the physically and operationally significant variability of K is greatest. The set of fields is represented in terms of a polar-stereographic grid-point array. It is quite feasible that the three-dimensional distribution of K, for an ocean region of interest, could be stored in the core memory of the CDC 6400 computer.

The interpolation, for location X_A, Y_A, z_A for example, may be performed by interpolating to X_A, Y_A at the four levels which include the depth z_A in the middle layer. Cubic interpolation in z would then complete the interpolation. It should be recalled that z_A may lie above the sea surface; in this case linear extrapolation of K from the surface and first sub-surface depth may be adequate.